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Effect of Dissipation on Non-Darcy convective Heat transfer in a Vertical Channel with Oscillatory Temperature

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1. INTRODUCTION

Flow through porous medium is very prevalent in nature and therefore the study of flow through porous medium has become of principal interesting in many scientific and engineering applications. In the theory of flow through a porous medium, the role of momentum equation or force balance is occupied by the numerous experimental observations summarized mathematically as the Darcy's law. It is observed that the Darcy's law is applicable as long as the Reynolds number based on average grain (pore) diameter does not exceed a value between 1 and 10. But in general the speed of specific discharge increases, the convective forces get developed and the internal stress generated in the fluid due to its viscous nature produces distortion in the velocity field. Also in the case of highly porous media such as fiber glass, papas dandelion etc., the viscous stress at the surface is able to penetrate into medium and produce gradient. Thus between the specific discharge and hydraulic gradient is inadequate in describing high speed flows or flows near surface which may be either permeable or not. Hence consideration for non-Darcian description for the viscous flow through a porous medium in warranted. Saffaman (18) employing statistical method derived governing equation for the flow in a porous medium which takes into account the viscous stress. Later another modification has been suggested by Brinkman (2)

$$0 = -\nabla p - (\frac{\mu}{k})\overline{v} + \mu \nabla^2 \overline{v}$$

in which $\mu \nabla^2 \overline{v}$ is intended to account for the distortion of the velocity profiles near the boundary. The same equation was derived analytically by Tam (24) to describe the viscous flow at low Reynolds number past a swam of small particles.

The process of free convection as a mode of heat transfer has wide applications in the fields of Chemical Engineering, Aeronautical and Nuclear power generation. It was shown by Gill and Casal (5) that the buoyancy significantly affects the flow of low Prandtal number fluids which is highly sensitive to gravitational force and the extent to which the buoyancy force influences a forced flow is a topic of interest. Free convection flows between two long vertical plates have been studied for many years because of their engineering applications in the fields of nuclear reactors, heat exchangers, cooling appliances in electronic instruments. These flows were studied by assuming the plates at two different constant temperatures or temperature of the plates varying linearly along the plates etc. The study of fully developed free convection flow between two parallel plates at constant temperature was initiated by Ostrach (13). Combined natural and forced convection laminar flow with linear wall temperature profile was also studied by Ostrach (14). The first exact solution for free convection in a vertical parallel plate channel with asymmetric heating for a fluid of constant properties was presented by Anug (1). Many of the early works on free convection flows in open channels have been reviewed by Manca *et al.* (7). Recently, Campo *et al.* (3) considered natural convection for heated iso-flux boundaries of the channel containing a low-Prandtl number fluid. Pantokratoras (15) studied the fully developed free convection flow between two asymmetrically heated vertical parallel plates for a fluid of varying thermo-physical properties. However, all the above

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studies are restricted to fully developed steady state flows. Very few papers deal with unsteady flow situations in vertical parallel plate channels. Transient free convection flow between two long vertical parallel plates maintained at constant but unequal temperatures was studied by Singh *et al.*(20). Jha *et al.* (6) extended the problem to consider symmetric heating of the channel walls. Narahari *et al*. (9) analyzed the transient free convection flow between two long vertical parallel plates with constant heat flux at one boundary, the other being maintained at a constant temperature. Singh and Paul (20) presented and analysis of the transient free convective flow of a viscous incompressible fluid between two parallel vertical walls occurring as a result of asymmetric heating / cooling of the walls. Narahari (10) presented an exact solution to the problem of unsteady free convective flow of a viscous incompressible fluid between two long vertical parallel plates with the plate temperature linearly varying with time at one boundary, that at the other boundary being held constant. There are many reasons for the flow to become unsteady. When the current is periodic due to on-off control mechanisms or due to partially rectified *a-c* voltage, there exist periodic heat inputs. Hence, it is important to study the effects of periodic heat flux on the unsteady natural convection, imposed on one of the plates of a channel formed by two long vertical parallel plates, the other being held at a constant initial fluid temperature. Recently Narahari(11)has discussed the unsteady free convection flow of dissipative viscous incompressible fluid between two long vertical parallel plates in which the temperature of one of the plates is oscillatory whereas that of the other plate is uniform.

Raptis and Singh (17) studied numerically the natural convection boundary layer flow past an impulsively started vertical plate in a Darcian porous medium. The thermal radiation effects on heat transfer in magneto-aerodynamic boundary layers has also received some attention, owing to astronautical re-entry, plasma flows in astrophysics, the planetary magneto-boundary layer and MHD propulsion systems. Mosa (8) discussed one of the first models for combined radiative hydromagnetic heat transfer, considered the case of free convective channel flows with an axial temperature gradient. Nath *et al.* (12) obtained a set of similarity solutions for radiative – MHD stellar point explosion dynamics using shooting methods

In this chapter we make an attempt to analyze the unsteady convective heat transfer of dissipative viscous fluid through a porous medium confined in a vertical channel on whose walls an oscillatory temperature is prescribed. Approximate solutions to coupled non-linear partial differential equations governing the flow and heat transfer are solved by a perturbation technique. The velocity, temperature, skin friction and rate of heat transfer are discussed for different variations of G, D^{-1},α and Ec.

2. FORMULATION AND SOLUTION OF THE PROBLEM

We consider the unsteady flow of a viscous incompressible fluid through a porous medium in a vertical channel bounded by flat walls in the presence of constant heat sources. The unsteadiness in the flow is due to the oscillatory temperature prescribed on the boundaries. We choose a Cartesian coordinate system 0(x y) with walls at y = ± 1 by using Boussinesq approximation we consider the density variation only on the buoyancy term. Also the kinematic viscosity, the thermal conductivity are treated as constants. The equations governing the flow and heat transfer are,

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} - (\frac{\mu}{k})u - (\frac{\sigma \mu_e^2 H_o^2}{\rho})u - \rho\, g \tag{2.1}$$

$$\rho_0 C_p \frac{\partial T}{\partial t} = K_f \frac{\partial^2 T}{\partial y^2} - Q(T - T_0) + 2\mu\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\mu}{k}u^2 \tag{2.2}$$

$$\rho - \rho_0 = -\beta_0(T - T_0) \tag{2.3}$$

where u is a velocity component in x-direction, T is a temperature, p is a pressure, ρ is a density ,k is the permeability of the porous medium,μ is dynamic viscosity, k_f is coefficient of thermal conductivity, β is coefficient of volume expansion and Q is the strength of heat source.

The boundary conditions are

$$\left.\begin{aligned} &u = 0,\quad T = T_1 \text{ at } y = -L \\ &u = 0,\ T = T_2 + \in(T_2 - T_1)\cos \omega t \end{aligned}\right\} \tag{2.7}$$

on introducing the non dimensional variables

$$u' = \frac{u}{\gamma / L}, \qquad y' = y/L,\ \theta = \frac{T - T_1}{T_2 - T_1},\ t' = \omega t,$$

Equations 2.1 & 2.2 reduce to (dropping the dashes)

$$\gamma^2 \frac{\partial u}{\partial t} = G\theta + \frac{\partial^2 u}{\partial y^2} - (D^{-1} + M^2)u \tag{2.8}$$

$$P\gamma^2 \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial y^2} - \alpha\,\theta + PEc\left(\frac{\partial u}{\partial y}\right)^2 + P\,EcD^{-1}u^2 \tag{2.9}$$

where

$G = \beta g L^3 \dfrac{(T_2 - T_1)}{\gamma^2}$ Grashof number)

$D^{-1} = \dfrac{L^2}{k}$ (Darcy parameter)

$P = \dfrac{\mu C_P}{K_f}$ (Prandtl number)

$\alpha = \dfrac{Q.L^2}{K_f}$ (Heat source parameter)

$Ec = \dfrac{\nu^2}{L^2\left(T - T_0\right)C_p}$ (Eckert Number)

$\gamma^2 = \dfrac{\omega L^2}{\nu}$ (Wormsely Number) $M_1^2 = M^2 + D^{-1}$

the transformed boundary conditions are

$$\left.\begin{array}{ll} u = 0, \quad \theta = 0, \quad \text{at } y = -1 \\ u = 0, \quad \theta = 1 + \in \cos(\omega t) \quad \text{at } y = +1 \end{array}\right\} \tag{2.10}$$

In view of the boundary conditions (2.10) we assume

$$\left.\begin{array}{l} u = u_0 + \in e^{it}\, u_1 \\ \theta = \theta_0 + \in e^{it}\, \theta_1 \end{array}\right\} \tag{2.11}$$

Substituting the series expansion (2.11) in equations (2.8) & (2.9) and separating the steady and transient terms we get

$$\frac{\partial^2 u_0}{\partial y^2} - M_1^2 u_0 = -G\theta_0 \tag{2.12}$$

$$\frac{\partial^2 u_1}{\partial y^2} - (M_1^2 + i\gamma^2)u_1 = -G\theta_1 \tag{2.13}$$

$$\frac{\partial^2 \theta_0}{\partial y^2} - \alpha\ \ \theta_0 + P\,Ec\frac{\partial^2 u_0}{\partial y^2} + P\,EcD^{-1}u_0^2 = 0 \tag{2.14}$$

$$\frac{\partial^2 \theta_1}{\partial y^2} - (\alpha\ \ + iP\gamma^2)\theta_1 + (2P\,Ec)\frac{\partial u_0}{\partial y}.\frac{\partial u_1}{\partial y} + (P\,EcD^{-1})u_0 u_1 \tag{2.15}$$

Since the equations (2.9 – 2.12) are non-linear coupled equations., assuming Ec << 1 we take

$$\left.\begin{array}{l} u_0 = u_{00} + Ec\, u_{01} \\ \theta_0 = \theta_{00} + Ec\, \theta_{01} \end{array}\right\} \tag{2.16}$$

Substituting (2.16) in equations (2.12 – 2.15) and separating the like terms we get

$$u_{00}^{11} - M_1^2 u_{00} = -G\theta_{00}, \qquad u_{00}(\pm 1) = 0 \qquad (2.17)$$

$$\theta_{00}^{11} - \alpha\,\theta_{00} = 0, \qquad \theta_{00}(-1) = 0,\ \theta_{00}(+1) = 1 \qquad (2.18)$$

$$u_{10}^{11} - (M_1^2 + i\gamma^2)u_{10} = -G\theta_{10}, \qquad u_{10}(\pm 1) = 0 \qquad (2.21)$$

$$\theta_{10}^{11} - iP\gamma^2\theta_{10} = 0, \qquad \theta_{10}(-1) = 0, \qquad \theta_{10}(+1) = 1 \qquad (2.22)$$

The solutions of equations (2.17)-(2.24) are

$$\theta_{00} = a_3(Ch(\beta_1 y) - Ch(\beta_1)\frac{Ch(\beta_2 y)}{Ch(\beta_2)}) + a_4(Sh(\beta_1 y) - Sh(\beta_1)\frac{Sh(\beta_2 y)}{Sh(\beta_2)}) + + 0.5(\frac{Sh(\beta_2 y)}{Sh(\beta_2)} + \frac{Ch(\beta_2 y)}{Ch(\beta_2)})$$

$$u_{00} = a_{11}(Sh(\beta_2 y) - Sh(\beta_2)\frac{Sh(M_1 y)}{Ch(M_1)}) + a_{12}(Ch(\beta_2 y) - Ch(\beta_2)\frac{Ch(M_1 y)}{Ch(M_1)}) + a_{13}(Ch(\beta_1 y) - Ch(\beta_1)\frac{Ch(M_1 y)}{Ch(M_1)}) + a_{14}(Sh(\beta_1 y) - Sh(\beta_1)\frac{Sh(M_1 y)}{Ch(M_1)})$$

$$u_{00} = a_{11}(Sh(\beta_2 y) - Sh(\beta_2)\frac{Sh(M_1 y)}{Ch(M_1)}) + a_{12}(Ch(\beta_2 y) - Ch(\beta_2)\frac{Ch(M_1 y)}{Ch(M_1)}) + a_{13}(Ch(\beta_1 y) - Ch(\beta_1)\frac{Ch(M_1 y)}{Ch(M_1)}) + a_{14}(Sh(\beta_1 y) - Sh(\beta_1)\frac{Sh(M_1 y)}{Ch(M_1)})$$

$$\theta_{10} = a_{107}(Ch(\beta_3 y) - Ch(\beta_3)\frac{Ch(\beta_4 y)}{Ch(\beta_4)}) + a_{108}(Sh(\beta_3 y) - Sh(\beta_3)\frac{Sh(\beta_4 y)}{Sh(\beta_4)}) + + 0.5(\frac{Ch(\beta_4 y)}{Ch(\beta_4)} + \frac{Sh(\beta_4 y)}{Sh(\beta_4)}$$

$$u_{10} = a_{109}(Ch(\beta_4 y) - Ch(\beta_4)\frac{Ch(\beta_5 y)}{Ch(\beta_5)}) + a_{110}(Sh(\beta_4 y) - Sh(\beta_4)\frac{Sh(\beta_5 y)}{Sh(\beta_5)}) +$$
$$+ a_{1111}(Ch(\beta_3 y) - Ch(\beta_3)\frac{Ch(\beta_5 y)}{Ch(\beta_5)}) + a_{112}(Sh(\beta_3 y) - Sh(\beta_3)\frac{Sh(\beta_5 y)}{Sh(\beta_5)})$$

Where,

$$\beta_1^2 = \gamma, \beta_2^2 = \alpha, \beta_3^2 = \gamma, \beta_4^2 = \alpha + iP\gamma_1^2, \beta_5^2 = M_1^2 + i\gamma^2,$$

$$k_1 = M_1 + \beta_2, k_2 = M_1 - \beta_2, k_3 = \beta_1 + \beta_2, k_4 = \beta_2 - \beta_1,$$

$$k_5 = \beta_1 + M_1, \quad k_6 = \beta_1 - M_1.$$

SHEAR STRESS and NUSSELT NUMBER

The Shear stress at the boundaries $\tau = \mu\left(\frac{du}{dy}\right)_{\pm L}$ which in the non dimensional form

reduces to $\tau^{\bullet} = \frac{\tau}{\left(\frac{\nu^2}{L^2}\right)} = \left(\frac{du}{dy}\right)_{y=\pm 1}$

and corresponding expressions are

$$\tau_{y=+1} = b_{54} + Ec\, b_{56} + \in e^{\gamma t} (b_{58} + Ec\, b_{60})$$

$$\tau_{y=+1} = b_{53} + Ec\, b_{55} + \in e^{\gamma t} (b_{57} + Ec\, b_{59})$$

The Rate of heat transfer (Nusselt number)at y = ± L is given by

$$q_w = Nu(\pm 1) = \left(\frac{d\theta}{dy}\right)_{y=\pm 1}$$

and the corresponding expressions are

$(Nu)_{y=-1} = b_{44} + Ec\, b_{46} + \in e^{it} (b_{48} + Ec\, b_{52})$

$(Nu)_{y=+1} = b_{43} + Ec\, b_{45} + \in e^{it} (b_{47} + Ec\, b_{51})$

Where $b_1, b_2, \ldots\ldots\ldots\ldots\ldots\ldots, b_{51}$ are constants given in the Appendix.

3. DISCUSSION OF THE NUMERICAL RESULTS

In this analysis we discuss the effect of dissipation on Non-Darcy convective heat transfer flow of a viscous electrically conducting fluid through porous medium in vertical channel with oscillatory temperature. We also consider heat generating sources in flow region. Through out the analysis we taken prandtle no. P=0.71.Figure 1-5 represent the axial velocity with different variation G, D^{-1}, M, α *and* E_c .Figure-1 represent variation of u with Grashof number G it is found that the axial velocity is in the vertically downward direction. $|u|$ Enhances increase in G>0 and depreciates with G<0 with maximum attend at y=0. Variation of u with D^{-1} shows that lesser the permeability of the porous medium larger the $|u|$ and further lowering of permeability smaller $|u|$ in entire flow region (fig-2). The variation of u with Hartmann number M shows that higher the Lorentz force (M≤4) larger $|u|$ and for further higher Lorentz force smaller $|u|$ in entire flow region (fig-3).An increase in strength of heat source parameter α leads to depreciates $|u|$ in flow region (fig-4). The variation of u with Ec shows that higher than dissipation heat lesser $|u|$ in the left half and larger $|u|$ in the right half of the channel (fig-5).

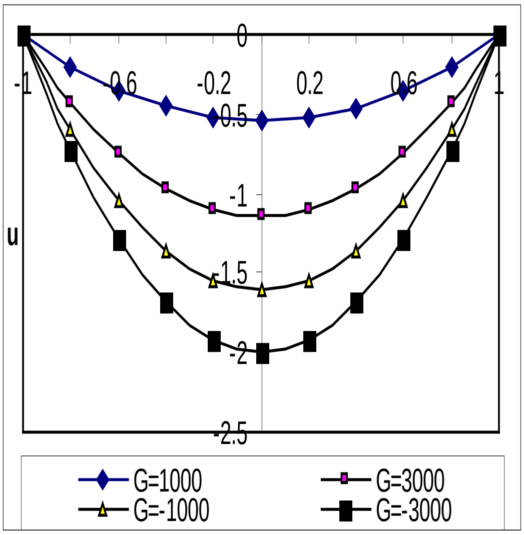

Fig. 1: Variation of u with G

	I	II	III	IV
G	10^3	$3x10^3$	-10^3	$-3x10^3$

$5x10^2$

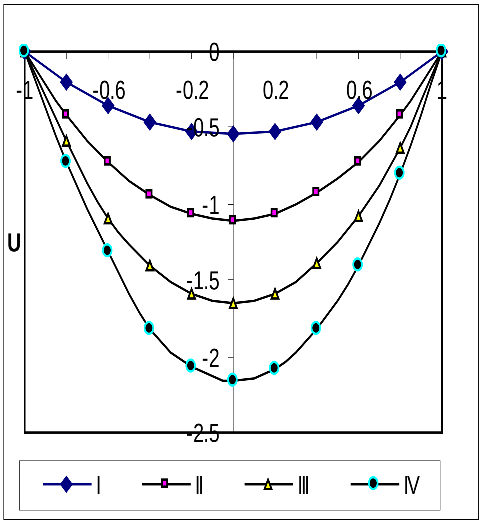

Fig. 2: Variation of u with D^{-1}

	I	II	III	IV
D^{-1}	10^2	$2X10^2$	$3X10^2$	

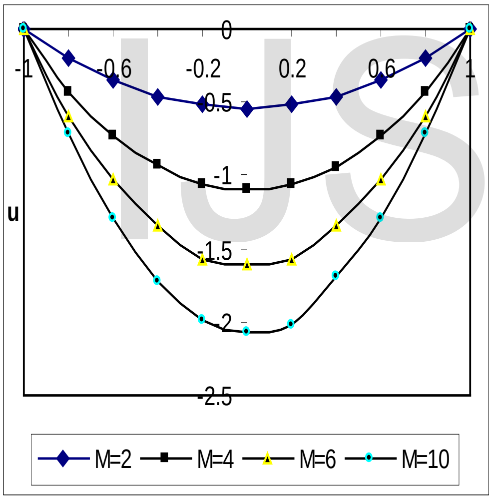

Fig. 3: Variation of u with M

	I	II	III	IV
M	2	4	6	10

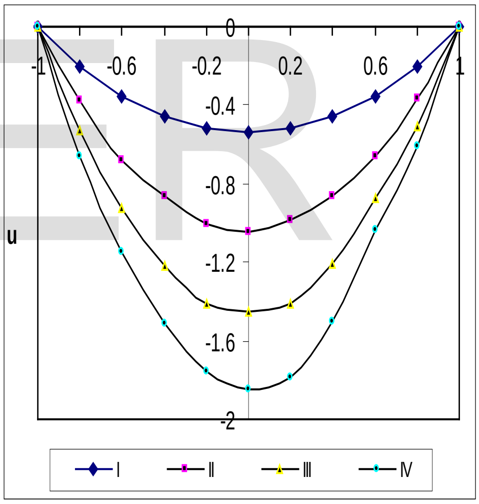

Fig. 4: Variation of u with α

	I	II	III	IV
α	2	4	6	10

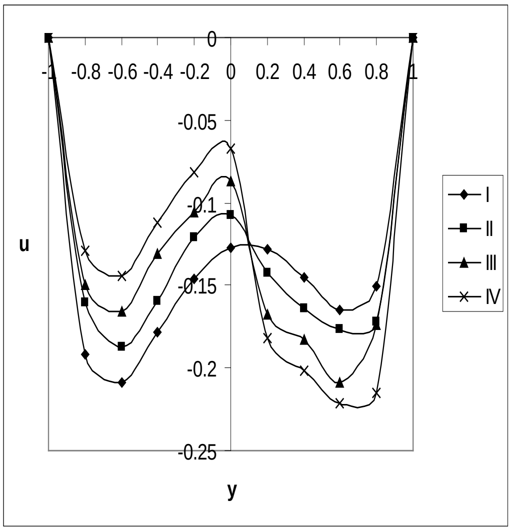

Fig. 5: Variation of u with Ec

	I	II	III	IV
Ec	0.001	0.003	0.005	0.007

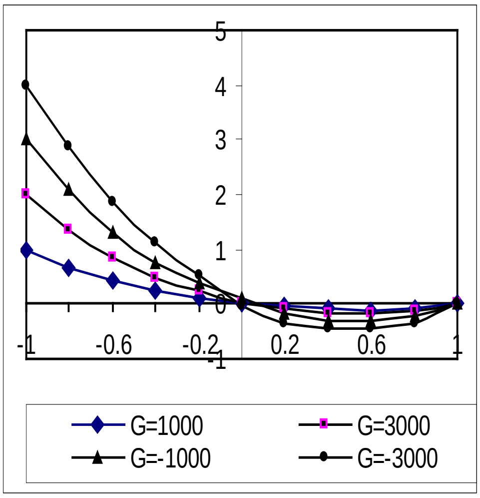

Fig. 6: Variation of with G

	I	II	III	IV
G	10^3	$3x10^3$	-10^3	$-3x10^3$

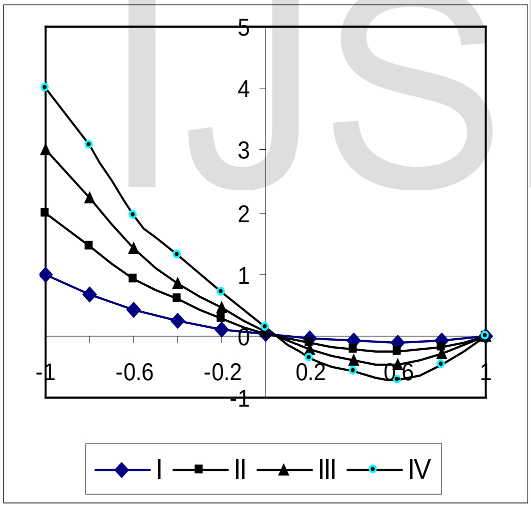

Fig. 7: Variation of u with D^{-1}

	I	II	III	IV
D^{-1}	10^2	$2X10^2$	$3X10^2$	$5x10^2$

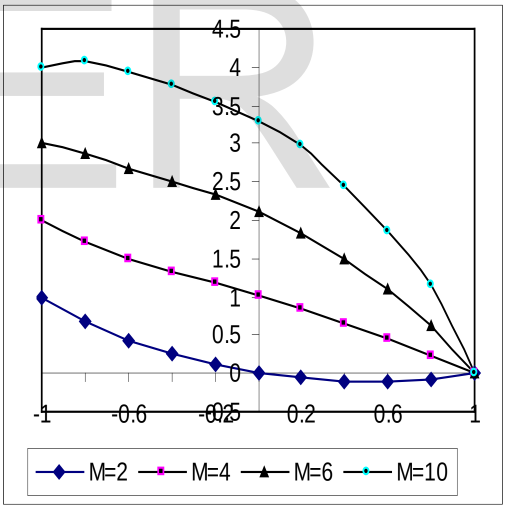

Fig. 8: Variation of u with M

	I	II	III	IV
M	2	4	6	10

The non dimensional temperature θ is shows in figures 6-10 for different parametric values. We follow the convection that the non dimension temperature θ is positive or negative according as the actual temperature is greater or lesser than T_2. Fig-6 represents the variation Groshof no. G it is found that the actual temperature depreciates with G> 0 and enhances with G<0. The variation of θ with $D^{-1}$and M shows that lesser the permeability of the porous medium larger the actual temperature in the flow region (fig-7) and the variation of θ with M shows that higher the Lorentz force larger the actual temperature in the flow region (fig-8). From (fig -9), we find that the actual temperature enhances $\alpha \le 4$ and higher $\alpha \ge 6$. It depreciates in entire flow region except in vicinity of y=-1 where it enhances with α. The variation of θ with Ec shows that higher the dissipative heat larger the actual temperature in the entire flow region (fig-10).

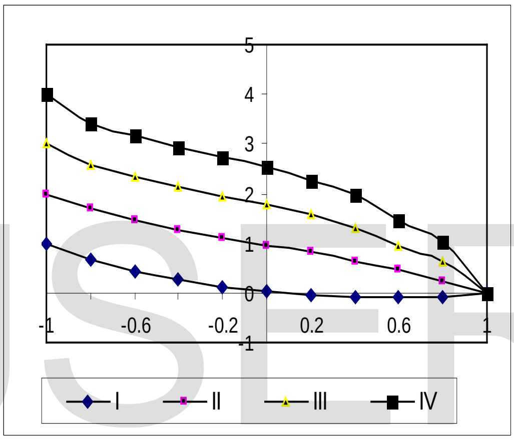

Fig. 9: Variation of u with α

	I	II	III	IV
α	2	4	6	10

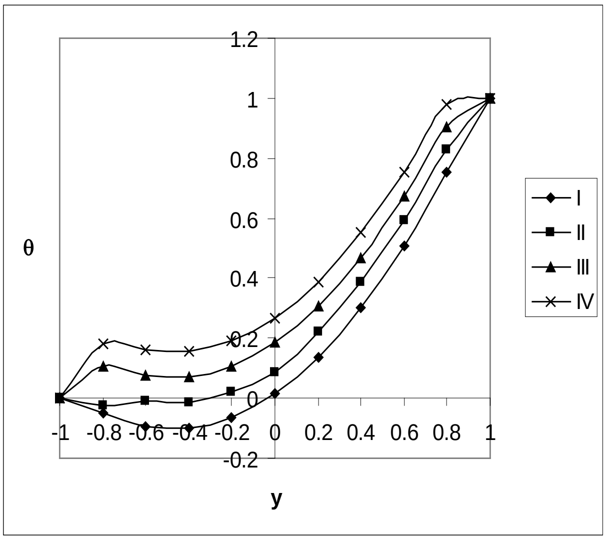

Fig.10: Variation of θ with Ec

	I	II	III	IV
Ec	0.001	0.003	0.005	0.007

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The shear stress τ at y=±1 is shown in the table 1-4 for different values G, D^{-1}, M, α and Ec. It is found that τ enhances with increase in $|G|$ at both the walls. The variation of τ with D^{-1} shows lesser the permeability of the porous medium larger $|\tau|$ for further lowering of the permeability smaller $|\tau|$ at both walls .Higher Lorentz force smaller τ and for further higher force larger θ at both wall y=±1. An increase in strength of heat source parameter α results an enhancement in θ at both the walls. The variation of τ with Ec shows that higher the dissipative heat larger τ at y=+1 and smaller at y=-1.

Table-1

Shear stress (τ) at y=+1

G	I	II	III	IV	V	VI
100	4.16028	4.2791	-0.3668	-5.1648	4.2642	4.5802
300	14.4808	14.838	0.8996	-6.4946	14.8642	15.1242
-100	-6.16088	-6.2791	-1.6332	5.1648	-6.3612	-6.6608
-300	-16.4808	-16.837	-2.8996	6.495	-17.124	-18.121

Annexure- 1

D	100	200	100	100	100	100
M	2	2	4	6	2	2
α	2	2	2	2	4	6

Table-2

Shear stress (τ) at y=-1

G	I	II	III	IV	V	VI
100	-3.941	-4.1842	0.10058	5.4949	-4.1242	-4.2408
300	-13.83	-14.557	-1.6983	6.4848	-14.098	-14.298
-100	5.9419	6.1842	1.8994	-4.495	6.1242	6.2383
-300	15.825	16.5526	3.6982	-5.485	16.4282	16.8542

See Annexure- 1

Table – 3

Shear stress (τ) at y = 1

G	I	II	III	IV
$1x10^3$	0.46187	0.46722	0.47223	0.49623
$3x10^3$	1.43906	1.56199	1.68387	1.80576
$-1x10^3$	-0.46187	-0.46722	-0.47223	-0.49623
$-3x10^3$	-1.43906	-1.56199	-1.68387	-1.80576
Ec	0.001	0.003	0.005	0.007

The shear stress τ at y=±1 is shown in the table 1-4 for different values G, D^{-1}, M, α and Ec. It is found that τ enhances with increase in $|G|$ at both the walls. The variation of τ with D^{-1} shows lesser the permeability of the porous medium larger $|\tau|$ for further lowering of the permeability smaller $|\tau|$ at both walls .Higher Lorentz force smaller τ and for further higher force larger θ at both wall y=±1. An increase in strength of heat source parameter α results an enhancement in θ at both the walls. The variation of τ with Ec shows that higher the dissipative heat larger τ at y=+1 and smaller at y=-1.

Table – 4

Shear stress (τ) at y = -1

G	I	II	III	IV
$1x10^3$	-1.04128	-1.03828	-1.03494	-1.03161
$3x10^3$	-3.08744	-3.00381	-2.91980	-2.83579
$-1x10^3$	1.04148	1.03828	1.03494	1.03161
$-3x10^3$	3.08744	3.00381	2.91980	2.83579
Ec	0.001	0.003	0.005	0.007

Table-5

Nusselt Number at y=+1

G	I	II	III	IV	V	VI
100	0.23091	0.22748	0.2212	0.2141	0.2542	0.2758
300	0.21491	0.20619	0.1906	0.1739	0.2259	0.2542
-100	0.2469	0.2487	0.2519	0.2543	0.2660	0.2889
-300	0.2629	0.2701	0.2826	0.2945	0.2809	0.2998

Annexure- 2

D	100	200	100	100	100	100
M	2	2	4	6	2	2
α	2	2	2	2	4	6

Table-6
Nusselt Number at y=-1

G	I	II	III	IV	V	VI
100	-1.30765	-1.2969	-1.2853	-1.2929	-1.2842	-1.2691
300	-1.2976	-1.2868	-1.2623	-1.2771	-1.2898	-1.2748
-100	-1.3176	-1.3268	-1.3329	-1.301	-1.3012	-1.2849
-300	-1.3376	-1.3468	-1.3642	-1.322	-1.3192	-1.3026

See Annexure- 2

The rate of heat transfer graduates the Nusselt number Nu shown in table 5-8 it is found that the rate of heat transfer reduces the increase in G>0 and enhances with G<0 at y=±1. The variation of Nu with Darcy parameter D^{-1}shows that lesser the permeability of the porous medium smaller the $|\mathrm{Nu}|$ in heating case and larger in cooling case. With respect to Hartmann number M shows higher the Lorentz force smaller $|\mathrm{Nu}|$ and for higher Lorentz force larger $|\mathrm{Nu}|$ at $y=\pm 1$.

Table – 7
Nusselt number (Nu) at y = 1

G	I	II	III	IV
$1X10^3$	-0.55027	-0.53007	-0.51297	-0.49587
$3x10^3$	-0.48588	-0.33687	-0.19097	-0.04507
Ec	0.001	0.003	0.005	0.007

Table – 8
Nusselt number (Nu) at y = -1

G	I	II	III	IV
10^3	1.55250	1.54583	1.53466	1.52350
$3X10^3$	1.50385	1.39987	1.29140	1.18294
Ec	0.001	0.003	0.005	0.007

In the heating case and in the cooling case it enhances Nu with M=4 depreciates with M=6 at y=+1 and at y=-1. An increase in strength of heat generating source parameter α enhances with radiation of heat transfer at y=+1 and depreciates it at y=-1. We find that rate of heat transfer with Ec is depreciates at both the walls.

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5. APPENDIX

Cosh( )=Ch( ) Sinh( ) =Sh( )

$M_1^2 = M^2 + D^{-1}$ $\beta_1^2 = iP\gamma^2$ $\beta_2^2 = M_1^2 + i\gamma^2$

$\beta_3 = M_1 + \beta_2$ $\beta_4 = M_1 - \beta_2$

$\beta_5 = M_1 + \beta_1$ $\beta_6 = M_1 - \beta_1$

$a_1 = \frac{-G\alpha}{2M_1^2}$ $a_2 = \frac{G}{2M_1^2}$

$a_3 = \frac{2G\alpha}{2M_1^4} + \frac{G(1+\alpha)}{2M_1^2}$ $a_4 = \frac{a_3 - a_1}{ch\,\mu_1}$

$a_5 = \frac{-a_2}{sh\,\mathrm{M}_1}$ $a_6 = \left(\frac{a_3 M}{ch\,\mathrm{M}_1} - M_1 a_1\right)$

$a_7 = 2a_1$ $a_8 = \frac{a_2 M_1}{sh\,\mathrm{M}_1}$

$a_9 = \left(\frac{a_3}{chM_1} - \frac{a_1}{ch\,\mathrm{M}_1}\right)$ $a_{10} = \frac{a_2}{sh\,\mathrm{M}_1}$

$$a_{11} = -P\frac{(a_6^2 + a_8^2 + a_2^2)}{2} - PD^{-1}\frac{(a_9^2 + a_{10}^2)}{2}$$

$$a_{12} = P\frac{(a_8^2 - a_6)}{2} - PD^{-1}\frac{(a_9^2 - a_{10}^2)}{2}$$

$a_{13} = [a_6 a_8 P + PD^{-1} a_9 a_{10}]$ $a_{14} = 2PD^{-1} a_1 a_9$

$a_{15} = 2PD^{-1} a_2 a_9 + 2Pa_7 a_8$ $a_{16} = 2PD^{-1} a_1 a_1$

$a_{17} = (2PD^{-1} a_2 a_{10} - 2Pa_6 a_7)$ $a_{18} = 2P\, a_2 a_8$

$a_{19} = 2P\, a_2 a_7$ $a_{20} = P\, a_7^2 + PD^{-1} a_2^2$

$a_{21} = 2PD^{-1} a_1 a_2$ $a_{22} = PD^{-1} a_1^2$

$a_{222} = 2Pa_2a_6$

$$a_{23} = \frac{a_{11}}{2}$$

$$a_{24} = \frac{a_{17}}{6} \qquad a_{25} = \frac{a_{20}}{12}$$

$$a_{26} = \frac{a_{21}}{20} \qquad a_{27} = \frac{a_{22}}{30}$$

$$a_{28} = \frac{a_{14}}{M_1} \qquad a_{29} = \frac{a_{16}}{M_1}$$

$$a_{30} = \frac{2a_{14}}{M_1^2} + \frac{a_{17}}{M_1} \qquad a_{31} = \frac{-a_{15}}{M_1} - \frac{2a_{16}}{M_1^2}$$

$$a_{32} = \frac{a_{15}}{M_1^2} - \frac{2a_{16}}{M_1^3} + \frac{a_{18}}{M_1^2} \qquad a_{33} = \frac{-a_{17}}{M_1^2} - \frac{a_{22}}{M_1^2} - \frac{2a_{14}}{M_1^3}$$

$$a_{34} = \frac{a_{12}}{4M_1^2} \qquad a_{35} = \frac{a_{13}}{4M_1^2}$$

$$a_{36} = ½ + a_{24} + a_{26} + a_{28}\, sh\, M_1 - a_{30}\, ch\, M_1 + a_{33}\, sh\, M_1 + a_{35}\, sh\,(2M_1)$$

$$a_{37} = ½ - a_{23} + a_{25} + a_{27} - a_{29}\, ch\, M_1 - a_{31}\, sh\, M_1 - a_{31}\, sh\,(2M_1) - a_{32}\, ch\, M_1 - a_{34}\, ch(2M_1)$$

$$a_{38} = \frac{Ga_{37}}{2M_1} \qquad a_{39} = \frac{Ga_{36}}{4M_1}$$

$$a_{40} = \frac{Ga_{35}}{3\mu_1^2} \qquad a_{41} = \frac{Ga_{34}}{3\mu_1^2}$$

$$a_{42} = \frac{Ga_{30}}{2\mu_1^2} - \frac{Ga_{33.}}{2\mu_1} \qquad a_{43} = \frac{Ga_{31}}{2\mu_1^2} - \frac{Ga_{32.}}{2\mu_1}$$

$$a_{44} = \frac{-Ga_{28}}{4M_1^2} - \frac{Ga_{30.}}{4M_1} \qquad a_{45} = \frac{Ga_{29}}{4M_1^2} - \frac{Ga_{31.}}{4M_1}$$

$$a_{46} = \frac{Ga_{28}}{6M_1} \qquad a_{47} = \frac{Ga_{29}}{6M_1}$$

$$a_{48} = -\left[\frac{\phi(+1) - \phi(-1)}{2\,\mathrm{ch}\,\mathrm{M}_1}\right] \qquad a_{49} = -\left[\frac{\phi(+1) - \phi(-1)}{2\,\mathrm{sh}\,\mathrm{M}_1}\right]$$

$$a_{50} = \frac{2}{2\,\mathrm{sh}\,M_1} \qquad a_{51} = \frac{2}{2\,\mathrm{ch}\,M_1}$$

$$a_{52} = \left[\frac{G}{2(\beta_1^2 - \beta_2^2)\,\mathrm{ch}\,\beta_1}\right] \qquad a_{53} = \frac{G}{2(\beta_1^2 - \beta_2^2)\,\mathrm{sh}\,\beta_1}$$

$$a_{54} = -\left[\frac{\phi(+1) + \phi(-1)}{2\,\mathrm{ch}\,\beta_2}\right] \qquad a_{55} = -\left[\frac{\phi(+1) + \phi(-1)}{2\,\mathrm{sh}\,\beta_2}\right]$$

$$a_{56} = -a_{52}\, ch\, \beta_1 \qquad a_{57} = -a_{53}\, sh\, \beta_1$$

$$a_{58} = \beta_1\, a_{52} \qquad a_{59} = a_{53}\beta_2 \frac{sh\,\beta_1}{sh\,\beta_2}$$

$$a_{60} = \beta_1 a_{53} \qquad a_{61} = a_4 M_1 \left(\frac{a_{57} + a_{58}}{2}\right)$$

$$a_{62} = a_4 M_1 \left(\frac{a_{58} - a_{59}}{2}\right) \qquad a_{63} = \frac{a_4 M_1}{2}\left(a_{59} + a_{57} + a_{68}\right)$$

$$a_{64} = \frac{a_4 M_1}{2}\left(a_{59} + a_{68} - a_{57}\right) \qquad a_{68} = 2a_1 a_{58}$$

$$a_{69} = 2a_1 a_{59} \qquad a_{70} = 2a_1 a_{60}$$

$$a_{71} = a_2 a_{57} \qquad a_{72} = a_2 a_{58}$$

$$a_{73} = a_2 a_{59} \qquad a_{74} = a_2 a_{60}$$

$$a_{75} = \frac{a_2\,\mathrm{ch}\,\beta_1}{ch\,\beta_2} \qquad a_{76} = \frac{a_{53}\,\mathrm{sh}\,\beta_3}{sh\,\beta_2}$$

$$a_{77} = \frac{a_4 a_{75}}{2} \qquad a_{78} = \frac{-a_4 a_{75}}{2}$$

$$a_{79} = a_4 \frac{(a_{76} - a_{53})}{2} \qquad a_{80} = \frac{-a_4}{2}(a_{76} + a_{53})$$

$$a_{82} = \frac{a_4}{2}(a_{53} - a_{76} + a_{53}) \qquad a_{83} = \frac{a_5}{2}(a_{52} + a_{75})$$

$$a_{84} = \frac{-a_5}{2}(a_{52} + a_{75}) \qquad a_{85} = a_1 a_{75}$$

$$a_{86} = a_2 a_{70} \qquad a_{87} = a_1 a_{52}$$

$a_{88} = a_1 a_{53}$

$a_{89} = a_2 a_{75}$

$a_{90} = a_2 a_{76}$

$a_{91} = a_1 a_{52}$

$a_{92} = a_2 a_{53}$

$a_{93} = a_3 a_{75}$

$a_{94} = a_3 a_{76}$

$a_{95} = a_3 a_{52}$

$a_{96} = a_3 a_{553}$

$$b_1 = \frac{a_{61} + a_{77}}{\beta_3^2 - \beta_1^2}$$

$$b_2 = \frac{a_{62} + a_{78}}{\beta_4^2 - \beta_1^2}$$

$$b_3 = \frac{a_{63} + a_{79}}{\beta_3^2 - \beta_1^2}$$

$$b_4 = \frac{a_{64} + a_{80}}{\beta_4^2 - \beta_1^2}$$

$$b_5 = \frac{a_{81} - a_{68}}{\beta_5^2 - \beta_1^2}$$

$$b_6 = \frac{a_{66} + a_{82}}{\beta_6^2 - \beta_1^2}$$

$$b_7 = \frac{a_{80}}{\beta_5^2 - \beta_1^2}$$

$$b_8 = \frac{a_{84}}{\beta_6^2 - \beta_1^2}$$

$$b_9 = \frac{a_{87}}{6\beta_1}$$

$$b_{10} = \frac{a_{88}}{6\beta_1}$$

$$b_{11} = \frac{a_{85}}{\beta_2^2 - \beta_1^2}$$

$$b_{12} = \frac{a_{86}}{\beta_2^2 - \beta_1^2}$$

$$b_{13} = \frac{\beta_1 a_{87} + (a_{68} - a_{92})}{4\beta_1^2}$$

$$b_{14} = \frac{\beta_1 a_{88} - (a_{70} + a_{91})}{4\beta_1^2}$$

$$b_{15} = \frac{4\beta_2 a_{85} - (\beta_2^2 - \beta_1^2)(a_{67} + a_{90})}{(\beta_2^2 - \beta_1^2)^2}$$

$$b_{16} = \frac{4\beta_2 a_{86} + (\beta_2^2 - \beta_1^2)(a_{69} + a_{89})}{(\beta_2^2 - \beta_1^2)^2}$$

$$b_{17} = \frac{a_{92} - a_{68} + \beta_1(a_{95} - a_{74})}{2\beta_1^2}$$

$$b_{18} = \frac{a_{70} + a_{91} + \beta(a_{96} - a_{72})}{2\beta_1^2}$$

$$b_{19} = \frac{2a_{85} + 2\beta_2(a_{67} + a_{90}) + (\beta_2^2 - \beta_1^2)(a_{73} - a_{93})}{(\beta_2^2 - \beta_1^2)^2}$$

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$$b_{20} = \frac{2a_{86} + \beta_2(a_{69} + a_{89}) + (\beta_2^2 - \beta_1^2)(a_{71} - a_{94})}{(\beta_2^2 - \beta_1^2)^2}$$

$$b_{211} = -\frac{\left(\phi_2(+1) + \phi_2(-1)\right)}{2\,\mathrm{ch}\,\beta_1} \qquad b_{222} = -\frac{\left(\phi_2(+1) - \phi_2(-1)\right)}{2\,\mathrm{sh}\,\beta_1}$$

$$b_{21} = \frac{+Gb_1}{\beta_3^2 - \beta_2^2} \qquad b_{22} = \frac{Gb_2}{\beta_4^2 - \beta_2^2}$$

$$b_{23} = \frac{Gb_3}{\beta_3^2 - \beta_2^2} \qquad b_{24} = \frac{Gb_4}{\beta_4^2 - \beta_2^2}$$

$$b_{25} = \frac{Gb_5}{\beta_5^2 - \beta_2^2} \qquad b_{26} = \frac{Gb_6}{\beta_6^2 - \beta_2^2}$$

$$b_{27} = \frac{Gb_7}{\beta_5^2 - \beta_2^2} \qquad b_{28} = \frac{Gb_8}{\beta_6^2 - \beta_2^2}$$

$$b_{29} = \frac{Gb_9}{\beta_1^2 - \beta_2^2} \qquad b_{30} = \frac{Gb_{10}}{\beta_1^2 - \beta_2^2}$$

$$b_{31} = \frac{-6\beta_1 Gb_{10} + (\beta_1^2 - \beta_2^2)Gb_{14}}{(\beta_1^2 - \beta_2^2)^2} \qquad b_{32} = \frac{-6\beta_1 Gb_9 - Gb_{13}}{(\beta_1^2 - \beta_2^2)^2}$$

$$b_{33} = \frac{Gb_{11}}{2\beta_2} \qquad b_{34} = \frac{Gb_{12}}{2\beta_2}$$

$$b_{35} = \frac{4Gb_{13}\beta_1 - 6Gb_9}{(\beta_1^2 - \beta_2^2)^2} \qquad b_{36} = \frac{4\beta_1 b_{14} G - 6Gb_{10}}{(\beta_1^2 - \beta_2^2)^2}$$

$$b_{37} = \frac{G(b_{12} - b_{16})}{2\beta_2^2} \qquad b_{38} = \frac{G(b_{11} - b_{15})}{2\beta_2^2}$$

$$b_{39} = \frac{Gb_{15}}{2\beta_2^2} \qquad b_{40} = \frac{Gb_{16}}{2\beta_2^2}$$

$$b_{41} = -\frac{\left[\phi_3(+1) + \phi_3(-1)\right]}{2\,\mathrm{ch}\,\beta_1} \qquad b_{42} = -\frac{\left[\phi_3(+1) + \phi_3(-1)\right]}{2\,\mathrm{sh}\,\beta_1}$$

$b_{43} = -\alpha + 0.5$ $\qquad b_{44} = \alpha + 0.5$

$b_{45} = 2a_{23} - 2a_{24} - 4a_{25} - 4a_{26} - 6a_{27} - a_{28}\,(\mathrm{sh}\,M_1 + M_1\,\mathrm{ch}\,M_1) + a_{29}\,(2\,\mathrm{ch}\,M_1 + M_1\,\mathrm{sh}\,M_1)$

$+ a_{30} M_1 \text{ sh } M_1 + a_{31} (2\text{ch } M_1 + M_1 \text{ sh } M_1) + a_{30} M_1 \text{ sh } M_1 + a_{31} (\text{sh}(M_1)+$

$+M_1 \ \text{ch}(M_1) – a_{32} M_1 \text{ sh}(M_1)– a_{33} (M_1 \text{ch}(M_1)– M_1 \text{ ch}M_1 –\text{sh } M_1) + 2M_1 a_{34} \text{ sh}$

$(2M_1) + a_{35} (2M_1 \text{ch } 2 M_1 – \text{sh } 2M_1) + 0.5$

$b_{46} = –2a_{23} – 2a_{24} + 4a_{25} – 4a_{26} + 6a_{27} – a_{28} (\text{sh}(M_1)+ M_1 \text{ ch}(M_1)) – a_{29} (2\text{ch}(M_1)+$

$+ M_1 \text{ sh}(M_1)) + a_{30} M_1 \text{ sh } M_1 – a_{31} (\text{sh}(M_1)+ M_1 \text{ ch}(M_1)) + a_{32} M_1 \text{ sh } (M_1)–$

$-a_{33} (M_1 \text{ ch } (M_1+) \text{ sh}(M_1)) – 2 M_1 a_{34} \text{ sh } (2M_1) + a_{35} (2M_1 \text{ ch } 2M_1 –\text{sh } 2M_1)$

$+ 0.5$

$b_{47} = 0.5 (\beta_1 \text{ Th } \beta_1 + \beta_1 \text{ ct h } \beta_1)$

$b_{48} = 0.5 (–\beta_1 \text{ Th } \beta_1 + \beta_1 \text{ ct h } \beta_1)$

$b_{49} = b_1\beta_3 \text{ sh } \beta_3 + b_2\beta_4 \text{ sh } \beta_4 + b_3\beta_3 \text{ ch } \beta_3 + b_4\beta_4 \text{ ch } \beta_4 – b_5\beta_5 \text{ sh } \beta_5 + b_6\beta_4 \text{ sh } \beta_6$

$+ b_7\beta_5 \text{ ch } \beta_5 + b_8\beta_6 \text{ ch } \beta_6 – b_9 (3 \text{ ch } \beta_1 + \beta_1 \text{ sh } \beta_1) – b_{10} (3 \text{ sh } \beta_1 + \beta_1 \text{ ch } \beta_1)$

$+ b_{11} (2 \text{ ch } \beta_2 + \beta_2 \text{ sh } \beta_2) + b_{12} (2 \text{ sh } \beta_2 + \beta_2 \text{ ch } \beta_2) + b_{13} (2 \text{ sh } \beta_1 + \beta_1 \text{ ch } \beta_1)$

$+ b_{14} (2 \text{ ch } \beta_1 + \beta_1 \text{ sh } \beta_1) + b_{15} (\text{sh } \beta_2 + \beta_2 \text{ ch } \beta_2) + b_{16} (\text{ch } \beta_2 + \beta_2 \text{ sh } \beta_2)$

$+ b_{17} (\text{ch } \beta_1 + \beta_1 \text{ sh } \beta_1) + b_{18} (\text{sh } \beta_1 + \beta_1 \text{ ch } \beta_1) + \beta_2 b_{19} \text{ sh } \beta_2 + b_{20} \beta_2 \text{ ch } \beta_2.$

$b_{50} = –b_1\beta_3 \text{ sh } \beta_3 – b_2\beta_4 \text{ sh } \beta_4 + b_3\beta_3 \text{ ch } \beta_3 + b_4\beta_4 \text{ ch } \beta_4 + b_5\beta_5 \text{ sh } \beta_5 + b_6\beta_6 \text{ sh } \beta_6$

$+ b_7\beta_5 \text{ ch } \beta_5 + b_8\beta_6 \text{ ch } \beta_6 – b_9 (3 \text{ ch } \beta_1 + \beta_1 \text{ sh } \beta_1) + b_{10} (3 \text{ sh } \beta_1 + \beta \text{ ch } \beta_1)$

$– b_{11} (2 \text{ ch } \beta_2 + \beta_2 \text{ sh } \beta_2) + b_{12} (2 \text{ sh } \beta_2 + \beta_2 \text{ ch } \beta_2) + b_{13} (2 \text{ sh } \beta_1 + \beta_1 \text{ ch } \beta_1)$

$– b_{14} (2 \text{ ch } \beta_1 + \beta_1 \text{ sh } \beta_1) – b_{15} (\text{sh } \beta_2 + \beta_2 \text{ ch } \beta_2) + b_{16} (\text{ch } \beta_2 + \beta_2 \text{ sh } \beta_2)$

$+ b_{17} (\text{ch } \beta_1 + \beta_1 \text{ sh } \beta_1) – b_{18} (\text{sh } \beta_1 + \beta_1 \text{ ch } \beta_1) – \beta_2 b_{19} \text{ sh } \beta_2 + b_{20} \beta_2 \text{ ch } \beta_2.$

$b_{51} = \beta_1 b_{21} \text{ sh } \beta_1 + \ \beta_1 b_{22} \text{ ch } \beta_1 + \phi^1(+1)$

$b_{52} = –\beta_1 b_{21} \text{ sh } \beta_1 + \ \beta_1 b_{22} \text{ ch } \beta_1 + \phi^1(–1)$

$b_{53} = a_3 M_1 \text{ Th } M_1 + a_1 (2+M_1 \text{ Th } M_1) + a_2 (1–M_1 \text{ cth } M_1)$

$b_{54} = –a_3 M_1 \text{ Th } M_1 + a_1 (2+M_1 \text{ Th } M_1) + a_2 (1–M_1 \text{ cth } M_1)$

$b_{55} = –a_{39} (2M_1 –\text{Th } M_1) – a_{41} (2M_1 \text{ sh } 2M_1 – M_1 \text{ ch } M_1 \text{ Th } M_1)$

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$+ a_{43} (\mathrm{sh}\, M_1 + M_1 \,\mathrm{ch}\, M_1 – M_1 \,\mathrm{sh}\, M_1 \mathrm{Th}\, M_1) + 2a_{45} \,\mathrm{ch}\, M_1 – a_{48} (1 – M_1 \,\mathrm{cth}\, M_1)$

$– a_{40} (2M_1 \,\mathrm{ch}\, 2M_1 – M_1 \,\mathrm{sh}\, 2M_1 \,\mathrm{cth}\, M_1) + a_{42} (\mathrm{ch}\, M_1 + M_1 \,\mathrm{sh}\, M_1 – M_1 \,\mathrm{ch}\, M_1 \,\mathrm{cth}\, M_1)$

$+ 2a_{44} \,\mathrm{sh}\, M_1 + a_{46} (3\mathrm{ch}\, M_1 + M_1 \,\mathrm{sh}\, M_1 – M_1 \,\mathrm{ch}\, M_1 \,\mathrm{cth}\, M_1).$

$b_{56} = a_{39} (2 – M_1 \mathrm{Th}\, M_1) + a_{41} (2M_1 \,\mathrm{sh}\, 2M_1 – M_1 \,\mathrm{ch}\, 2 M_1 \,\mathrm{Th}\, M_1)$

$– a_{43} (\mathrm{sh}\, M_1 + M_1 \,\mathrm{ch}\, M_1 – M_1 \,\mathrm{sh}\, M_1 \mathrm{Th}\, M_1) – 2a_{45} \,\mathrm{ch}\, M_1 – a_{38} (1 + M_1 \,\mathrm{cth}\, M_1)$

$– a_{40} (2M_1 \,\mathrm{ch}\, 2M_1 – M_1 \,\mathrm{sh}\, 2M_1 \,\mathrm{cth}\, M_1) + a_{42} (\mathrm{ch}\, M_1 + M_1 \,\mathrm{sh}\, M_1 – M_1 \,\mathrm{ch}\, M_1 \,\mathrm{cth} M_1)$

$+ 2a_{44} \,\mathrm{sh}\, M_1 + a_{46} (3\mathrm{ch}\, M_1 + M_1 \,\mathrm{sh}\, M_1 – M_1 \,\mathrm{ch}\, M_1 \,\mathrm{cth}\, M_1)$

$b_{57} = a_{52} (–\beta_2 \,\mathrm{Th}\, \beta_2 \,\mathrm{ch}\, \beta_1 + \beta_1 \,\mathrm{sh}\, \beta_1) + a_{53} (\beta_2 \,\mathrm{cth}\, \beta_2 \,\mathrm{sh}\, \beta_1 – \beta_1 \,\mathrm{ch}\, \beta_1)$

$b_{58} = a_{52} (–\beta_2 \,\mathrm{Th}\, \beta_2 \,\mathrm{ch}\, \beta_1 + \beta_1 \,\mathrm{sh}\, \beta_1) + a_{53} (\beta_2 \,\mathrm{cth}\, \beta_2 \,\mathrm{sh}\, \beta_1 – \beta_1 \,\mathrm{ch}\, \beta_1)$

$b_{59} = \beta_2 b_{41} \,\mathrm{sh}\, \beta_2 + \beta_2 b_{42} \,\mathrm{ch}\, \beta_2 + \phi_3^1(+1)$

$b_{60} = –\beta_2 b_{41} \,\mathrm{sh}\, \beta_2 + \beta_2 b_{42} \,\mathrm{ch}\, \beta_2 + \phi_3^1(–1)$

$b_{61} = –b_{21}\beta_3 \,\mathrm{sh}\, \beta_3 – b_{22}\beta_4 \,\mathrm{sh}\, \beta_4 – \beta_3 b_{23} \,\mathrm{ch}\, \beta_3 – \beta_4 b_{24} \,\mathrm{ch}\, \beta_4 + \beta_5 b_{25} \,\mathrm{sh}\, \beta_5$

$– \beta_6 b_{26} \,\mathrm{sh}\, \beta_6 – \beta_5 b_{27} \,\mathrm{ch}\, \beta_5 – \beta_6 b_{28} \,\mathrm{ch}\, \beta_6 + b_{29}(3 \,\mathrm{ch}\, \beta_1 + \beta_1 \mathrm{sh}\, \beta_1)$

$+ b_{30}(3 \,\mathrm{sh}\, \beta_1 + \beta_1 \,\mathrm{ch}\, \beta_1) + b_{31}(2 \,\mathrm{ch}\, \beta_1 + \beta_1 \mathrm{sh}\, \beta_1) + b_{32}(2 \,\mathrm{sh}\, \beta_1 + \beta_1 \,\mathrm{ch}\, \beta_1)$

$– b_{33}(2 \,\mathrm{sh}\, \beta_2 + \beta_2 \,\mathrm{ch}\, \beta_2) – b_{34}(2 \,\mathrm{ch}\, \beta_2 + \beta_2 \,\mathrm{sh}\, \beta_2) + b_{35}(\mathrm{ch}\, \beta_1 + \beta_1 \,\mathrm{sh}\, \beta_1)$

$+ b_{36}(\mathrm{sh}\, \beta_1 + \beta_1 \,\mathrm{ch}\, \beta_1) + b_{37}(\mathrm{sh}\, \beta_2 + \beta_2 \,\mathrm{ch}\, \beta_2) + b_{38}(\mathrm{ch}\, \beta_2 + \beta_2 \,\mathrm{sh}\, \beta_2)$

$+ b_{39}\ \beta_2 \,\mathrm{ch}\, \beta_2 + b_{40}\beta_2 \,\mathrm{sh}\, \beta_2.$

$b_{62} = b_{21}\beta_3 \,\mathrm{sh}\, \beta_3 + b_4\beta_{22} \,\mathrm{sh}\, \beta_2 – \beta_3 b_{23} \,\mathrm{ch}\, \beta_3 – \beta_4 b_{24} \,\mathrm{ch}\, \beta_4 + \beta_5 b_{25} \,\mathrm{ch}\, \beta_5$

$– \beta_6 b_{26} \,\mathrm{sh}\, \beta_6 – \beta_5 b_{27} \,\mathrm{ch}\, \beta_5 – \beta_6 b_{28} \,\mathrm{ch}\, \beta_6 + b_{29}(3 \,\mathrm{ch}\, \beta_1 + \beta_1 \mathrm{sh}\, \beta_1)$

$– b_{30}(3 \,\mathrm{sh}\, \beta_1 + \beta_1 \,\mathrm{ch}\, \beta_1) – b_{31}(2 \,\mathrm{ch}\, \beta_1 + \beta_1 \mathrm{sh}\, \beta_1) + b_{32}(2 \,\mathrm{sh}\, \beta_1 + \beta_1 \,\mathrm{ch}\, \beta_1)$

$– b_{33}(2 \,\mathrm{sh}\, \beta_2 + \beta_2 \,\mathrm{ch}\, \beta_2) + b_{34}(2 \,\mathrm{ch}\, \beta_2 + \beta_2 \,\mathrm{sh}\, \beta_2) + b_{35}(\mathrm{ch}\, \beta_1 + \beta_1 \,\mathrm{sh}\, \beta_1)$

$– b_{36}(\mathrm{sh}\, \beta_1 + \beta_1 \,\mathrm{ch}\, \beta_1) – b_{37}(\mathrm{sh}\, \beta_2 + \beta_2 \,\mathrm{ch}\, \beta_2) + b_{38}(\mathrm{ch}\, \beta_2 + \beta_2 \,\mathrm{sh}\, \beta_2)$

$+ b_{39}\ \beta_2 \,\mathrm{ch}\, \beta_2 + b_{40}\beta_2 \,\mathrm{sh}\, \beta_2.$